August 2001, Day 1, Question 9

Let A and B be $n \times n$ matrices.

(a) Show that the following is not true:

$$\frac{d}{dt}e^{A+tB} = e^{A+tB}B.$$

(b) Show that

$$\frac{d}{dt}\mathrm{Tr}\left(e^{A+tB}\right) = \mathrm{Tr}\left(e^{A+tB}B\right).$$

I have not found a completely convincing argument for part (a) other than an explicit example. But I have a proof of part (b) that exploits the fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Note that this implies

$$\begin{aligned}\mathrm{Tr}\,(ABA) &= \mathrm{Tr}\,(A \cdot BA) \\ &= \mathrm{Tr}\,(BA \cdot A) \\ &= \mathrm{Tr}\left(BA^2\right),\end{aligned}$$

with similar results holding for arbitrary products of A and B in arbitrary order. The prood also exploits the linearity of trace.

Before we continue, it should be noted that if we have three distinct $n \times n$ matrices A, B, and C, the trace does have some order-dependence. For example,

$$\begin{aligned}\mathrm{Tr}\,(ABC) &= \mathrm{Tr}\,(AB \cdot C) \\ &= \mathrm{Tr}\,(CAB) \\ &= \mathrm{Tr}\,(CA \cdot B) \\ &= \mathrm{Tr}\,(BCA).\end{aligned}$$

Hence $\mathrm{Tr}\,(ABC)$ is equal to the trace of any cyclic permutation of A, B, and C, but $\mathrm{Tr}\,(ABC)$ is not necessarily equal to $\mathrm{Tr}\,(ACB)$.

Now we start the proof.

$$\begin{aligned}e^{A+tB} &= \sum_{n=0}^{\infty} \frac{1}{n!}\,(A+tB)^n \\ \mathrm{Tr}\left(e^{A+tB}\right) &= \mathrm{Tr}\left(\sum_{n=0}^{\infty} \frac{1}{n!}\,(A+tB)^n\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\mathrm{Tr}\,((A+tB)^n)\end{aligned}$$

$(A+tB)^n$ is a sum of products of A and B in a variety of orders, so the binomial formula does not hold for $(A+tB)^n$. But the trace of a product of $n-k$ As and

k Bs, in any order, is equal to $\mathrm{Tr}\left(A^{n-k}B^k\right)$ by the fact noted above. We use this below.

$$\begin{aligned}
\mathrm{Tr}\left((A+tB)^n\right) &= \sum_{k=0}^{n}\binom{n}{k}\mathrm{Tr}\left(A^{n-k}t^kB^k\right)\\
&= \sum_{k=0}^{n}\binom{n}{k}t^k\mathrm{Tr}\left(A^{n-k}B^k\right)\\
\frac{d}{dt}\mathrm{Tr}\left((A+tB)^n\right) &= \sum_{k=0}^{n}\binom{n}{k}kt^{k-1}\mathrm{Tr}\left(A^{n-k}B^k\right)\\
&= \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}kt^{k-1}\mathrm{Tr}\left(A^{n-k}B^k\right)\\
&= \sum_{k=1}^{n}\frac{n!}{(k-1)!(n-k)!}t^{k-1}\mathrm{Tr}\left(A^{n-k}B^k\right)
\end{aligned}$$

Let $\ell = k-1$. Then $k = \ell+1$, and we have

$$\begin{aligned}
\sum_{k=1}^{n}\frac{n!}{(k-1)!(n-k)!}t^{k-1}\mathrm{Tr}\left(A^{n-k}B^k\right) &= \sum_{\ell+1=1}^{n}\frac{n!}{\ell!(n-\ell-1)!}t^{\ell}\mathrm{Tr}\left(A^{n-\ell-1}B^{\ell+1}\right)\\
&= \sum_{\ell=0}^{n}\frac{n(n-1)!}{\ell!(n-\ell-1)!}\mathrm{Tr}\left(A^{n-\ell-1}t^{\ell}B^{\ell}B\right)\\
&= \mathrm{Tr}\left(n\sum_{\ell=0}^{n}\frac{(n-1)!}{\ell!(n-\ell-1)!}\mathrm{Tr}\left(A^{n-\ell-1}t^{\ell}B^{\ell}B\right)\right)\\
&= \mathrm{Tr}\left(n\left(A+tB\right)^{n-1}B\right),
\end{aligned}$$

where we have used the linearity of trace and its other property in the last step.

This last result is plugged into the earlier equation.

$$\begin{aligned}
\frac{d}{dt}\mathrm{Tr}\left(e^{A+tB}\right) &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d}{dt}\mathrm{Tr}\left((A+tB)^n\right)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}\left(n\left(A+tB\right)^{n-1}B\right)\\
&= \mathrm{Tr}\left(\sum_{n=0}^{\infty}\frac{n}{n!}\left(A+tB\right)^{n-1}B\right)\\
&= \mathrm{Tr}\left(\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(A+tB\right)^{n-1}B\right)\\
&= \mathrm{Tr}\left(\sum_{m=0}^{\infty}\frac{1}{m!}\left(A+tB\right)^{m}B\right) \qquad (m=n-1)\\
&= \mathrm{Tr}\left(e^{A+tB}B\right)
\end{aligned}$$